

Fully Homomorphic Encryption

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Fully homomorphic encryption

- Multiplicatively homomorphic: RSA.

$$\begin{aligned} c_1 &= m_1^e \bmod N \\ c_2 &= m_2^e \bmod N \end{aligned} \Rightarrow c_1 \cdot c_2 = (m_1 \cdot m_2)^e \bmod N$$

- Additively homomorphic: Paillier

$$\begin{aligned} c_1 &= g^{m_1} \bmod N^2 \\ c_2 &= g^{m_2} \bmod N^2 \end{aligned} \Rightarrow c_1 \cdot c_2 = g^{m_1+m_2} \bmod N^2$$

- Fully homomorphic: homomorphic for both addition and multiplication
 - Open problem until Gentry's breakthrough in 2009.

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Fully homomorphic public-key encryption

- We restrict ourselves to public-key encryption of a single bit:
 - $0 \rightarrow 203ef6124 \dots 23ab87_{16}$
 - $1 \rightarrow b327653c1 \dots db3265_{16}$
- Fully homomorphic property
 - Given $E(b_0)$ and $E(b_1)$, one can compute $E(b_0 \oplus b_1)$ and $E(b_0 \cdot b_1)$ without knowing the private-key.
- Computing over a ring:
 - Given a circuit with xors and ands, and encrypted input bits, one can compute the output in encrypted form, without knowing the private key.
 - As a result: publicly compute any function on encrypted data (or at least any function that can be represented as a boolean circuit with polynomially many gates).

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What fully homomorphic encryption brings you

- You have a software that given the revenue, past income, headcount, etc., of a company can predict its future stock price.
 - I want to know the future stock price of my company, but I don't want to disclose confidential information.
 - And you don't want to give me your software containing secret formulas.
- Using homomorphic encryption:
 - I encrypt all the inputs using fully homomorphic encryption and send them to you in encrypted form.
 - You process all my inputs, viewing your software as a circuit.
 - You send me the result, still encrypted.
 - I decrypt the result and get the predicted stock price.
 - You didn't learn any information about my company.
- More generally:
 - Cool buzzwords like **secure cloud computing**.
 - Cool mathematical challenges.

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Fully Homomorphic Encryption Schemes

- 1. Breakthrough scheme of Gentry [G09], based on ideal lattices.
 - Some optimizations by [SV10].
 - Implementation [GH11]: PK size: 2.3 GB, recrypt: 30 min.
- 2. van Dijk, Gentry, Halevi and Vaikuntanathan's scheme over the integers [DGHV10].

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Gentry's technique

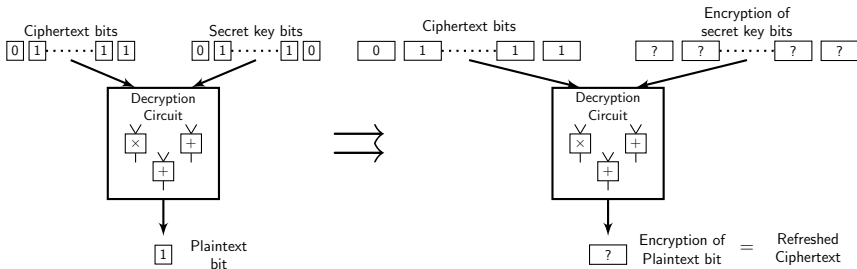
- To build a FHE scheme, start from the **somewhat homomorphic** scheme, that is:
 - Only a polynomial of small degree can be homomorphically applied on ciphertexts.
 - Otherwise the noise becomes too large and decryption becomes incorrect.
- Then, “squash” the decryption procedure:
 - express the decryption function as a low degree polynomial in the bits of the ciphertext c and the secret key sk (equivalently a boolean circuit of small depth).

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Ciphertext refresh

- Gentry's breakthrough idea: refresh the ciphertext using the decryption circuit homomorphically.
 - Evaluate the decryption polynomial not on the bits of the ciphertext c and the secret key sk , but homomorphically on the **encryption** of those bits.
 - Instead of recovering the bit plaintext m , one gets an encryption of this bit plaintext, *i.e.* yet another ciphertext for the same plaintext.



Ciphertext refresh

- Refreshed ciphertext:
 - If the degree of the decryption polynomial is small enough, the resulting noise in this new ciphertext can be smaller than in the original ciphertext
- Fully homomorphic encryption:
 - Given two refreshed ciphertexts one can apply again the homomorphic operation (either addition or multiplication), which was not necessarily possible on the original ciphertexts because of the noise threshold.
 - Using this “ciphertext refresh” procedure the number of homomorphic operations becomes unlimited and we get a fully homomorphic encryption scheme.

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The DGHV Scheme

- Ciphertext for $m \in \{0, 1\}$:

$$c = q \cdot p + 2r + m$$

where p is the secret-key, q and r are randoms.

- Decryption:

$$(c \bmod p) \bmod 2 = m$$

- Parameters:



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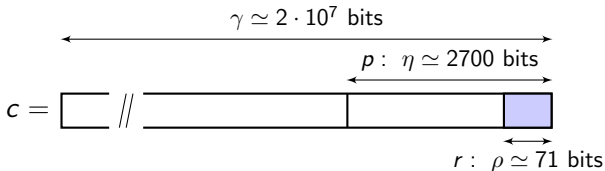
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Homomorphic Properties of DGHV

- Addition:

$$\begin{aligned}c_1 &= q_1 \cdot p + 2r_1 + m_1 \\c_2 &= q_2 \cdot p + 2r_2 + m_2\end{aligned} \Rightarrow c_1 + c_2 = q' \cdot p + 2r' + m_1 + m_2$$

- Multiplication:

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with

$$r'' = 2r_1r_2 + r_1m_2 + r_2m_1$$

- Noise becomes twice larger.

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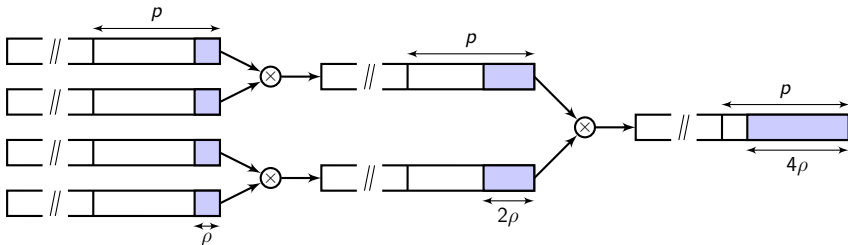
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Somewhat homomorphic scheme

- The number of multiplications is limited.
 - Noise grows with the number of multiplications.
 - Noise must remain $< p$ for correct decryption.



Public-key Encryption with DGHV

- Ciphertext

$$c = q \cdot p + 2r + m$$

- Public-key: a set of τ encryptions of 0's.

$$x_i = q_i \cdot p + 2r_i$$

- Public-key encryption:

$$c = m + 2r + \sum_{i=1}^{\tau} \varepsilon_i \cdot x_i$$

for random $\varepsilon_i \in \{0, 1\}$.

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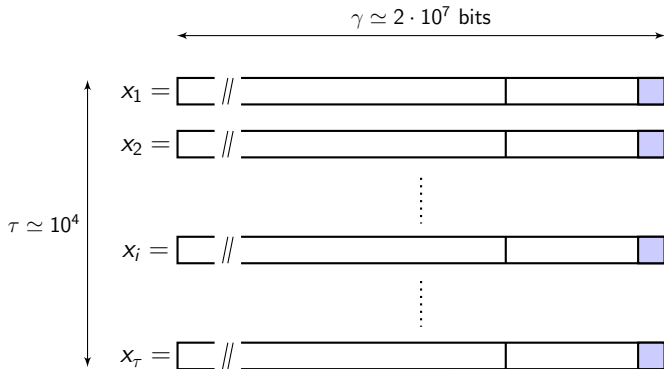
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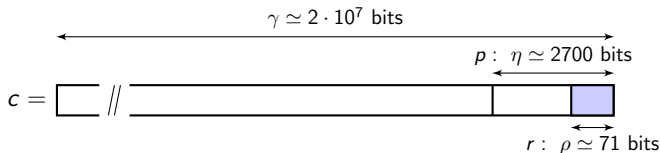
Public Key Size



- Public-key size: $\tau \cdot \gamma = 2 \cdot 10^{11}$ bits = 25 GB !
 - In [CMNT11], with quadratic encryption, PK size of 1 GB.

DGHV Ciphertext Compression

- Ciphertext: $c = q \cdot p + 2r + m$



- Compute a pseudo-random $\chi = f(\text{seed})$ of γ bits.

$$\chi = \boxed{} // \overline{}$$

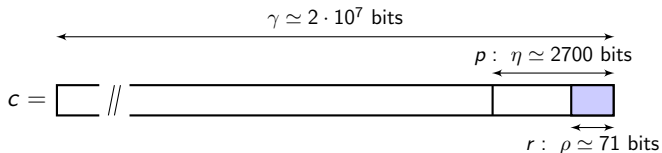
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- Storage:** ≈ 2700 bits instead of $2 \cdot 10^7$ bits !

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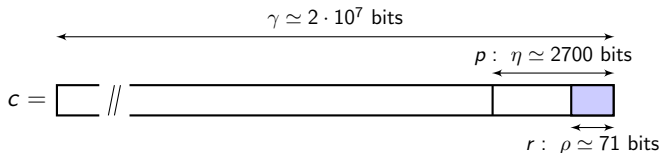
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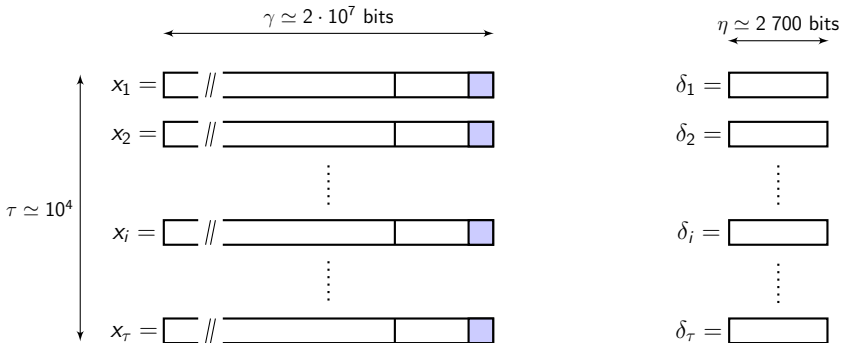
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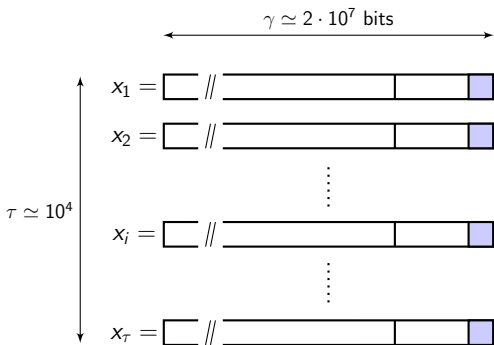
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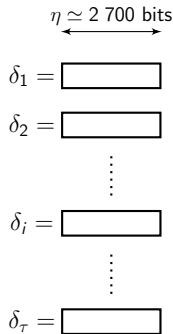
Compressed Public Key



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Old PK: 25 GB



New PK: 3.4 MB !

Security of Compressed PK

- Original DGHV scheme is semantically secure, under the approximate-gcd assumption.
 - Approximate-gcd problem: given a set of $x_i = q_i \cdot p + r_i$, recover p .
- Compressed public key
 - *seed* is part of the public-key, to recover the x_i 's, so we cannot argue that $f(\text{seed})$ is pseudo-random.
 - Security in the random oracle model only, still based on approximate-gcd.

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PK Generation

$$\begin{array}{l} \chi_i = H(seed, i) \\ \delta_i = [\chi_i]_p + \lambda_i \cdot p - r_i \\ \downarrow \\ x_i = \chi_i - \delta_i \end{array}$$

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Simulation in ROM

$$\begin{array}{l} \uparrow \\ H(\text{seed}, i) \leftarrow x_i + \delta_i \\ \delta_i \leftarrow \{0, 1\}^{\eta+\lambda} \\ x_i = q_i \cdot p + r_i \end{array}$$

Hardness assumption for semantic security

- Original DGHV scheme: secure under the **General Approximate Common Divisor** (GACD) assumption.
 - Given polynomially many $x_i = p \cdot q_i + r_i$, find p .
- Efficient DGHV variant: secure under the **Partial Approximate Common Divisor** (PACD) assumption.
 - Given $x_0 = p \cdot q_0$ and polynomially many $x_i = p \cdot q_i + r_i$, find p .
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 - How much easier ?

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Brute force attack on the noise

- Brute force attack on the noise.
 - Given $x_0 = q_0 \cdot p$ and $x_1 = q_1 \cdot p + r_1$ with $|r_1| < 2^\rho$, one can guess r_1 and compute $\gcd(x_0, x_1 - r_1)$ to recover p .
 - Requires 2^ρ gcd computation
- Countermeasure:
 - Take a sufficiently large ρ

Solving PACD

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- Variant suggested by Phong Nguyen, still in $\mathcal{O}(2^\rho)$:

$$p = \gcd \left(x_0, \prod_{i=0}^{2^\rho-1} (x_1 - i) \bmod x_0 \right)$$

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Approximate GCD attack

- Consider t integers: $x_i = p \cdot q_i + r_i$ and $x_0 = p \cdot q_0$.
 - Consider a vector \vec{u} orthogonal to the x_i 's:

$$\sum_{i=1}^t u_i \cdot x_i = 0 \pmod{x_0}$$

- This gives $\sum_{i=1}^t u_i \cdot r_i = 0 \pmod{p}$.
- If the u_i 's are sufficiently small, since the r_i 's are small this equality will hold over \mathbb{Z} .
 - Such vector \vec{u} can be found using LLL.
- By collecting many orthogonal vectors one can recover \vec{r} and eventually the secret key p
- Countermeasure
 - The size γ of the x_i 's must be sufficiently large.

The DGHV scheme (simplified)

- Key generation:

- Generate a set of τ public integers:

$$x_i = p \cdot q_i + r_i, \quad 1 \leq i \leq \tau$$

and $x_0 = p \cdot q_0$, where p is a secret prime.

- Size of p is η . Size of x_i is γ . Size of r_i is ρ .
- Encryption of a message $m \in \{0, 1\}$:
 - Choose a random subset $S \subset \{1, 2, \dots, \tau\}$ and a random integer r in $(-2^{\rho'}, 2^{\rho'})$, and output the ciphertext:

$$c = m + 2r + 2 \sum_{i \in S} x_i \pmod{x_0}$$

- Decryption:

$$c \equiv m + 2r + 2 \sum_{i \in S} r_i \pmod{p}$$

- Output $m \leftarrow (c \pmod{p}) \pmod{2}$

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The DGHV scheme (contd.)

- Noise in ciphertext:
 - $c = m + 2 \cdot r' \pmod p$ where $r' = r + \sum_{i \in S} r_i$
 - r' is the noise in the ciphertext.
 - It must remain $< p$ for correct decryption.
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 - $c_1 + c_2 = m_1 + m_2 + 2(r'_1 + r'_2) \pmod p$
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- Somewhat homomorphic scheme
 - Noise grows with every homomorphic addition or multiplication.
 - A limited number of homomorphic operations is supported.
 - This limits the degree of the polynomial that can be applied on ciphertexts.

The DGHV scheme (contd.)

- Noise in ciphertext:
 - $c = m + 2 \cdot r' \pmod p$ where $r' = r + \sum_{i \in S} r_i$
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Parameter estimates

Security parameter λ .

- ρ : size of noise should be λ bits
- ρ' : size of secondary noise 2λ bits
- η : size of p , $\approx \lambda^2$ bits
- γ : size of x_i , $\approx \lambda^5$
- τ : number of elements (x_i 's) in the public key, $\gamma + \lambda$

Public key size $\approx \gamma^2 \approx \lambda^{10}$ ($\approx 2^{62}$ bits for $\lambda = 72$ bits of security).

Constraints on the parameters

- ρ : size of noise
 - $\rho \geq 37$ bits to avoid brute-force attack on the noise
- η : size of p
 - The squashed scheme has a decryption polynomial of degree 30.
 - We must allow for an additional multiplication, so degree $d = 60$
 - $\eta \geq (d + 8)\rho = 2516$ bits.
- γ : size of x_i :
 - $\gamma \simeq 12 \cdot 10^6$ bits to avoid lattice attacks
- Public-key size:
 - If we take $\tau = \gamma$, we get a pk size of $\tau \cdot \gamma = \gamma^2 = 1.4 \cdot 10^{14}$ bits.
 - Initial scheme unpractical.

The squashed scheme from DGHV

- The basic decryption $m \leftarrow (c \bmod p) \bmod 2$ cannot be directly expressed as a boolean circuit of low depth.
- But it can be written as:

$$m \leftarrow [c]_2 \oplus [[c \cdot (1/p)]]_2$$

and this formula can be used for ciphertext refresh if $1/p$ can be put in a compact encrypted form in the public key.

- Idea (Gentry, DGHV): use secret sharing. Represent $1/p$ as a sparse subset sum:

$$[2^\kappa/p] = \sum_{i=1}^{\oplus} s_i \cdot u_i$$

with random κ -bit integers u_i , and $s_i \in \{0, 1\}$. Publish the u_i 's and encryptions of the s_i 's.

- The decryption function can then be expressed as a polynomial of low degree (30) in the s_i 's.

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Compressing the public-key

- Size of public-key
 - In the squashed scheme, $\Theta = \tilde{O}(\lambda^5)$ additional elements y_i in the public key, each of size $\kappa = \tilde{O}(\lambda^5)$ bits.
 - Therefore this gives again a $\tilde{O}(\lambda^{10})$ -bit public key, instead of $\tilde{O}(\lambda^5)$ in our variant.
- Using a pseudo-random number generator:
 - Generate $\Theta - 1$ random integers $u_i \in [0, 2^{\kappa+1})$ for $2 \leq i \leq \Theta$, using a pseudo-random generator $f(\text{se})$ where the seed se is generated at random during key generation and made part of the public key.
 - Take $s_1 = 1$ and generate u_1 so that

$$\sum_{i \in S} u_i = x_p \pmod{2^{\kappa+1}}$$

The decryption circuit

- We must compute:

$$m \leftarrow c^* - \left[\sum_{i=1}^{\Theta} s_i \cdot z_i \right] \pmod 2$$

- Trick from Gentry-Halevi:
 - Split the Θ secret key bits into θ boxes of size $B = \Theta/\theta$ each.
 - Then only one secret key bit inside every box is equal to one

- New decryption formula: $m \leftarrow c^* - \left[\sum_{k=1}^{\theta} \left(\sum_{i=1}^B s_{k,i} z_{k,i} \right) \right]_2$

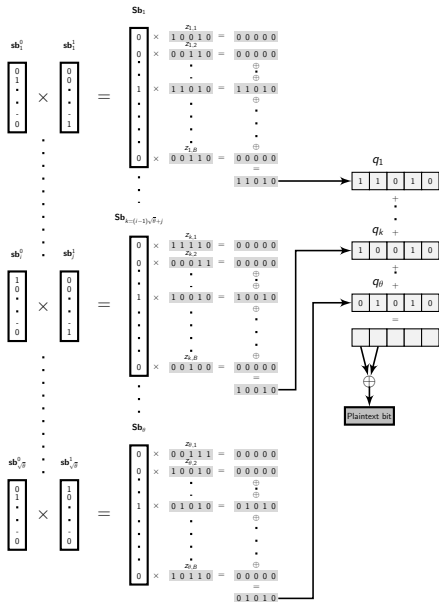
- The sum:

$$q_k \stackrel{\text{def}}{=} \sum_{i=1}^B s_{k,i} z_{k,i}$$

is obtained by adding B numbers, only one being non-zero.

- To compute the j -th bit of q_k it suffices to xor all the j -th bits of the numbers $s_{k,i} \cdot z_{k,i}$.

The decryption circuit

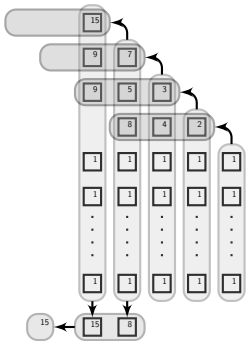


Grade School addition

- The decryption equation is now:

$$m \leftarrow c^* - \left[\sum_{k=1}^{\theta} q_k \right] \pmod{2}$$

- where the q_k 's are rational in $[0, 2)$ with n bits of precision after the binary point.



Gentry's Bootstrapping

- To refresh a ciphertext:
 - Publicly evaluate the decryption circuit homomorphically on a ciphertext
 - This produces another ciphertext for the same message, but with possibly less noise (a “recryption”).
- Encryption of the secret-key bits are required:
 - The vector \vec{s} has $\Theta = \tilde{O}(\lambda^5)$ bits.
 - $\Theta = \tilde{O}(\lambda^5)$ ciphertexts, each of size $\gamma = \tilde{O}(\lambda^5)$, give again a public key size of $\tilde{O}(\lambda^{10})$.

PK size and timings

Instance	λ	ρ	η	γ	pk size	Recrypt
Toy	42	27	1026	$150 \cdot 10^3$	77 KB	0.41 s
Small	52	41	1558	$830 \cdot 10^3$	437 KB	4.5 s
Medium	62	56	2128	$4.2 \cdot 10^6$	2.2 MB	51 s
Large	72	71	2698	$19 \cdot 10^6$	10.3 MB	11 min

- Updated parameters to take into account the Chen-Nguyen attack.
- PK size: **10.3 MB** instead of 1 GB in [CMNT11].