Cryptography

Course 2: attacks against RSA

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Attacks against RSA

- Factoring
  - Equivalence between factoring and breaking RSA ?
- Mathematical attacks
  - Attacks against plain RSA encryption and signature
  - Heuristic countermeasures
  - Low private / public exponent attacks
  - Provably secure constructions
- Implementation attacks
  - Timing attacks, power attacks and fault attacks
  - Countermeasures

- Key generation:
  - Generate two large distinct primes p and q of same bit-size.
  - Compute $n = p \cdot q$ and $\phi = (p-1)(q-1)$.
  - Select a random integer e, $1 < e < \phi$ such that $\gcd(e, \phi) = 1$
  - Compute the unique integer d such that
$$e \cdot d \equiv 1 \mod \phi$$
    using the extended Euclidean algorithm.
  - The public key is (n, e). The private key is d.

RSA encryption

- Encryption
  - Given a message $m \in [0, n-1]$ and the recipent's public-key (n, e), compute the ciphertext:

$$c = m^e \mod n$$

- Decryption
  - Given a ciphertext c, to recover m, compute:

$$m = c^d \mod n$$

Low private exponent attacks

- To reduce decryption time, one could use a small d
  - Wiener's attack: recover d if $d < N^{0.25}$
- Boneh and Durfee's attack (1999)
  - Recover d if $d < N^{0.29}$
  - Based on lattice reduction and Coppersmith's technique
  - Open problem: extend to $d < N^{0.5}$
- Conclusion: devastating attack
  - Use a full-size d

Low public exponent attack

- To reduce encryption time, one can use a small e
  - For example $e = 3$ or $e = 2^{16} + 1$
- Coppersmith's theorem :
  - Let N be an integer and f be a polynomial of degree δ. Given N and f, one can recover in polynomial time all x_0 such that $f(x_0) = 0 \mod N$ and $x_0 < N^{1/\delta}$.
- Application: partially known message attack :
  - If $c = (B\|m)^3 \mod N$, one can recover m if $|m| < |N|/3$
  - Define $f(x) = (B \cdot 2^k + x)^3 - c \mod N$.
  - Then $f(m) = 0 \mod N$ and apply Coppersmith's theorem to recover m.

Low public exponent attack

- Coppersmith's short pad attack
  - Let $c_1 = (m\|r_1)^3 \mod N$ and $c_2 = (m\|r_2)^3 \mod N$
  - One can recover m if $r_1, r_2 < N^{1/9}$
  - Let $g_1(x, y) = x^3 - c_1$ and $g_2(x, y) = (x + y)^3 - c_2$.
  - g_1 and g_2 have a common root $(m\|r_1, r_2 - r_1)$ modulo N.
  - $h(y) = \mathrm{Res}_x(g_1, g_2)$ has a root $\Delta = r_2 - r_1$, with $\deg h = 9$.
  - To recover $m\|r_1$, take gcd of $g_1(x, \Delta)$ and $g_2(x, \Delta)$.
- Conclusion:
  - Attack only works for particular encryption schemes.
  - Low public exponent is secure when provably secure construction is used. One often takes $e = 2^{16} + 1$.

Solving Modular polynomial equations

- Solving $p(x) = 0 \mod N$ when N is of unknown factorization: hard problem.
  - For $p(x) = x^2 - a$, equivalent to factoring N.
  - For $p(x) = x^e - a$, equivalent to inverting RSA.
- Coppersmith showed (E96) that finding small roots is easy.
  - When $\deg p = \delta$, finds in polynomial time all integer x_0 such that $p(x_0) = 0 \mod N$ and $|x_0| \leq N^{1/\delta}$.
  - Based the LLL lattice reduction algorithm.
- Can be heuristically extended to more variables.

Applications in cryptography

- Coppersmith's algorithm has numerous applications in cryptanalysis :
  - Cryptanalysis of plain RSA when some part of the message is known :
    - If $c = (B + x_0)^3 \mod N$, let $p(x) = (B + x)^3 - c$ and recover x_0 if $x_0 < N^{1/3}$.
  - Factoring $n = p^r q$ for large r (Boneh and al., C99).
- Applications in provable security :
  - Improved security proof for RSA-OAEP with low-exponent (Shoup, C01).

Solving $p(x) = 0 \mod N$

- Find a small linear integer combination $h(x)$ of the polynomials :
  - $q_{ik}(x) = x^i \cdot N^{\ell-k} p^k(x) \mod N^\ell$
  - For some ℓ and $0 \leq i < \delta$ and $0 \leq k \leq \ell$.
  - $p(x_0) = 0 \mod N \Rightarrow p^k(x_0) = 0 \mod N^k \Rightarrow q_{ik}(x_0) = 0 \mod N^\ell$.
  - Then $h(x_0) = 0 \mod N^\ell$.
- If the coefficients of $h(x)$ are small enough :
  - Then $h(x_0) = 0$ holds over $\mathbb{Z}$.
  - x_0 can be found using any standard root-finding algorithm.

Solving $x^2 + ax + b = 0 \mod N$.

- Illustration with a polynomial of degree 2 :
  - Let $p(x) = x^2 + ax + b \mod N$.
  - We must find x_0 such that $p(x_0) = 0 \mod N$ and $|x_0| \leq X$.
- We are interested in finding a small linear integer combination of the polynomials :
  - $p(x)$, Nx and N.
  - Then $h(x_0) = 0 \mod N$.
- If the coefficients of $h(x)$ are small enough :
  - Then $h(x_0) = 0$ also holds over $\mathbb{Z}$,
  - which enables to recover x_0.

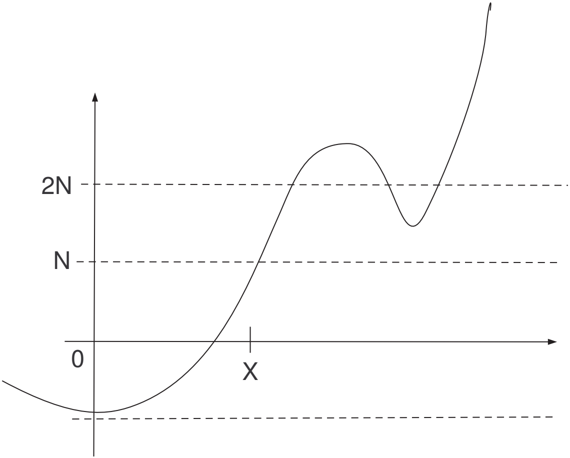
Howgrave-Graham lemma

- Given $h(x) = \sum h_i x^i$, let $\|h\|^2 = \sum h_i^2$.
- Howgrave-Graham lemma :
  - Let $h \in \mathbb{Z}[x]$ be a sum of at most ω monomials. If $h(x_0) = 0 \mod N$ with $|x_0| \leq X$ and $\|h(xX)\| < N/\sqrt{\omega}$, then $h(x_0) = 0$ holds over $\mathbb{Z}$.
  - Proof :

$$\begin{aligned} |h(x_0)| &= \left|\sum h_i x_0^i\right| = \left|\sum h_i X^i \left(\frac{x_0}{X}\right)^i\right| \\ &\leq \sum \left|h_i X^i \left(\frac{x_0}{X}\right)^i\right| \leq \sum \left|h_i X^i\right| \\ &\leq \sqrt{\omega}\|h(xX)\| < N \end{aligned}$$

Since $h(x_0) = 0 \mod N$, this gives $h(x_0) = 0$.

Illustration of HG lemma



HG lemma

- The coefficients of $h(xX)$ must be small:
  - $h(xX)$ is a linear integer combination of the polynomials

$$\begin{aligned} p(xX) &= X^2 \cdot x^2 + aX \cdot x + b \\ q_1(xX) &= NX \cdot x \\ q_2(xX) &= N \end{aligned}$$

- We must find a small integer linear combination of the vectors:
  - $[X^2, aX, b]$, $[0, NX, 0]$ and $[0, 0, N]$
- Tool: LLL algorithm.

Lattice and lattice reduction

- We must find a small linear integer combination $h(xX)$ of the polynomials $p(xX)$, xXN and N.
  - Let L be the corresponding lattice, with a basis of row vectors :
$$\begin{bmatrix} X^2 & aX & b \\ & NX & \\ & & N \end{bmatrix}$$
  - Using LLL, one can find a lattice vector b of norm : $\|b\| \leq 2(\det L)^{1/3} \leq 2N^{2/3}X$
- Then if $X < N^{1/3}/4$, then $\|h(xX)\| = \|b\| < N/2$
  - Howgrave-Graham lemma applies and $h(x_0) = 0$.

Lattice

- Definition :
  - Let $u_1, \ldots, u_\omega \in \mathbb{Z}^n$ be linearly independent vectors with $\omega \leq n$. The lattice L spanned by the u_i's is
    $L = \left\{ \sum_{i=1}^{\omega} n_i \cdot u_i \middle| \, n_i \in \mathbb{Z} \right\}$
  - If L is full rank ($\omega = n$), then $\det L = |\det M|$, where M is the matrix whose rows are the basis vectors $u_1, \ldots, u_\omega$.
- The LLL algorithm :
  - The LLL algorithm, given $(u_1, \ldots, u_\omega)$, finds in polynomial time a vector b_1 such that:

$$\|b_1\| \leq 2^{(\omega-1)/4} \det(L)^{1/\omega}$$

Solving $p(x) = 0 \mod N$

- The previous bound gives $|x_0| \leq N^{1/3}/4$.
  - But Coppersmith's bound gives $|x_0| \leq N^{1/2}$.
- Technique : work modulo N^k instead of N.
  - Let $q(x) = (p(x))^2$. Then $q(x_0) = 0 \mod N^2$.
  - $q(x) = x^4 + a'x^3 + b'x^2 + c'x + d'$.
  - Find a small linear combination $h(x)$ of the polynomials $q(x)$, $Nxp(x)$, $Np(x)$, N^2x and N^2.
  - Then $h(x_0) = 0 \mod N^2$.
  - If the coefficients of $h(x)$ are small enough, then $h(x_0) = 0$.

Details when working modulo N^2

- Lattice basis :

$$\begin{bmatrix} X^4 & a'X^3 & b'X^2 & c'X & d' \\ & NX^3 & NaX^2 & NbX & \\ & & NX^2 & NaX & Nb \\ & & & N^2X & \\ & & & & N^2 \end{bmatrix}$$

- Using LLL, one gets :
  - $\|h(xX)\| \leq 2 \cdot (\det L)^{1/5} \leq 2X^2N^{6/5}$
  - If $X \leq N^{2/5}/6$, then $\|h(xX)\| \leq N^2/3$ and $h(x_0) = 0$.