# Introduction to Cryptography

## Part 2: public-key cryptography

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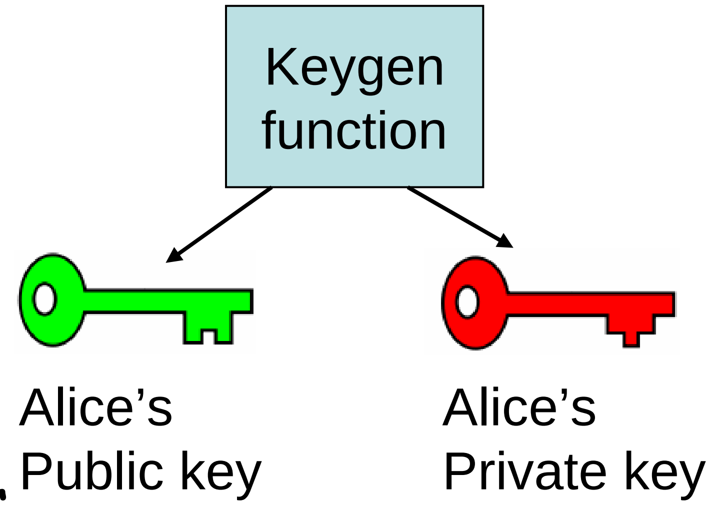
# Public-key cryptography

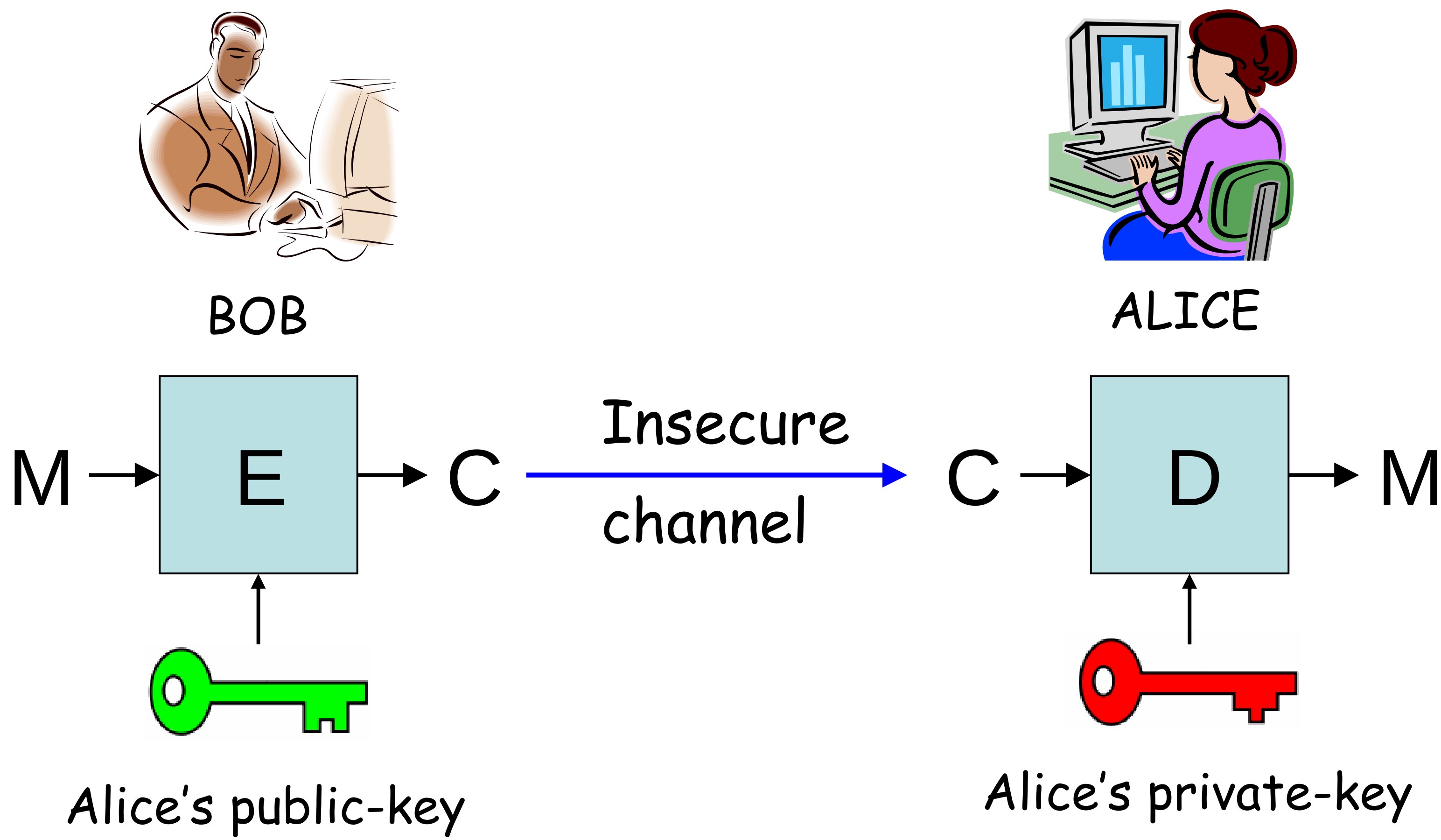
- Invented by Diffie and Hellman in 1976. Revolutionized the field.
- Each user now has two keys
  - A public key
  - A private key
- Should be hard to compute the private key from the public key.
- Enables:
  - Asymmetric encryption
  - Digital signatures
  - Key exchange
  - Identification,

  and many other functionalities



# Public-key encryption

- Solves the key distribution issue



# RSA

- Invented by Rivest, Shamir and Adleman in 1977.
- Still the most widely used PK algorithm.
- Public key: $n=p.q$ and $e$
  - Primes $p$ and $q$ remain secret.
- Private key: $d$ such that

$$e.d=1 \bmod (p-1)(q-1)$$

# RSA

- Encryption using public $n$,$e$:

  $c=m^e \bmod n$

- Decryption using private $d$:

  $m=c^d \bmod n$

- Decryption works because:

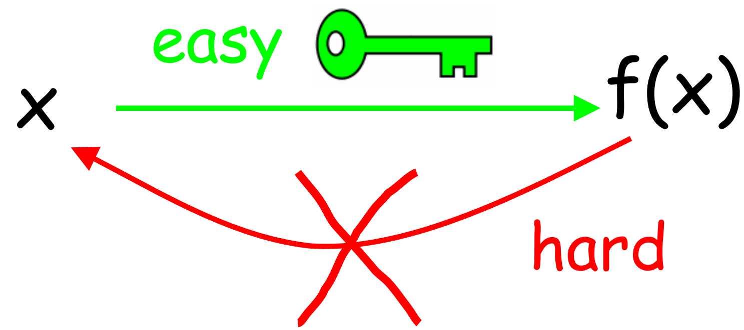
  $m=c^d=(m^e)^d=m^{e.d}=m$

  because:

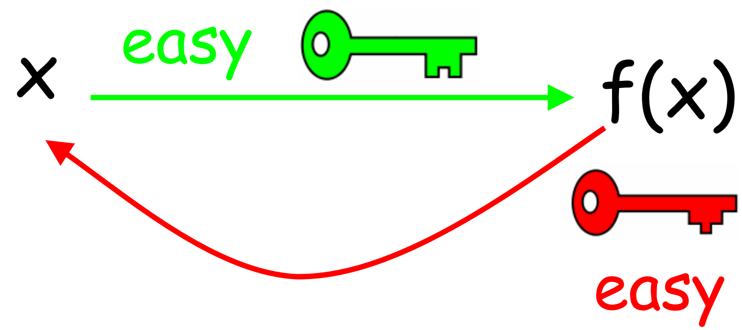
  $e.d=1 \bmod f$

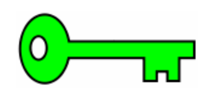
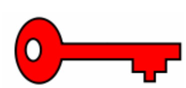
# RSA: trapdoor one-way permutation

- Trapdoor unknown:



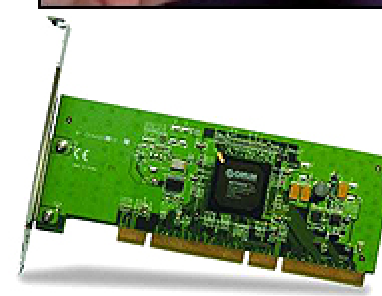

- Trapdoor known:




- Asymmetric encryption:
  - Everybody can encrypt to Alice using 
  - Only Alice can decrypt using 

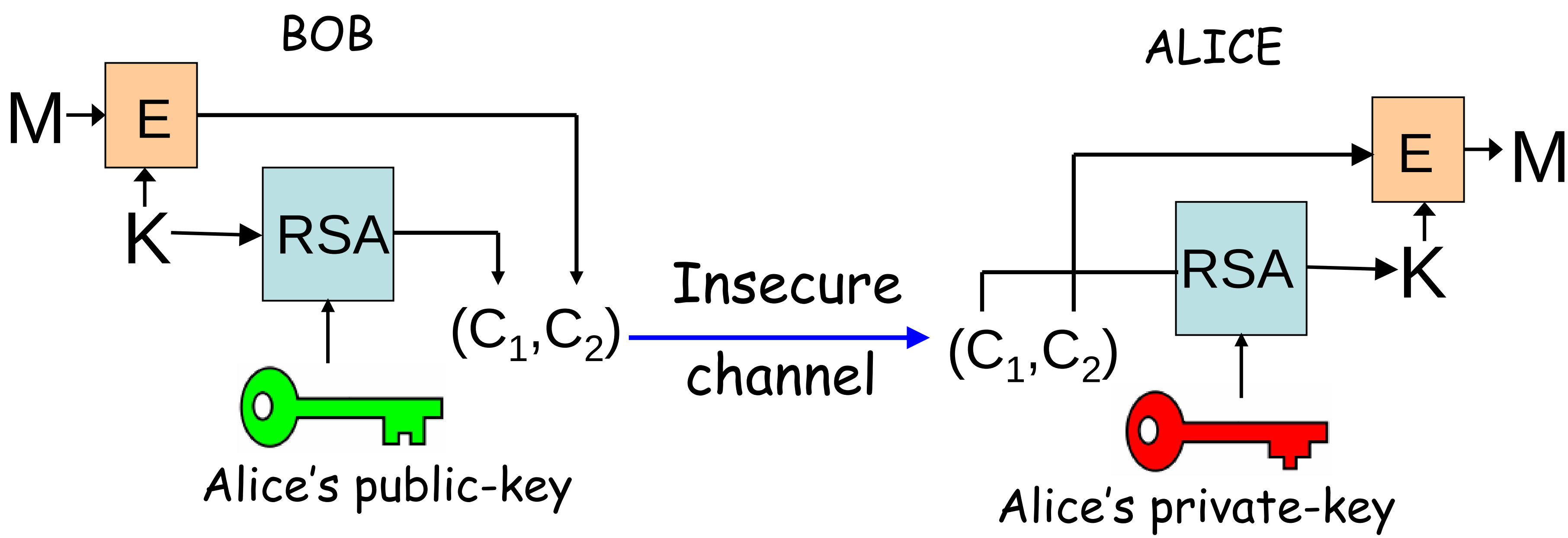
# Implementation of RSA

- Required: computing with large integers
  - more than 1024 bits.
- In software
  - big integer library: GMP, NTL
- In hardware
  - Cryptoprocessor for smart-card
  - Hardware accelerator for PC.



# Speed of RSA

- RSA much slower than AES and other secret key algorithms.
  - to encrypt long messages, encrypt a symmetric key K with RSA, and encrypt the long message with K.



# Security of RSA

- Security of RSA is based on the hardness of factorization
  - Given $n=p.q$, no known efficient algorithm to recover $p$ and $q$.
  - Factorization record: 663 bits (2005)
- Public modulus $n$ must be large enough
  - At least 1024 bits. 2048 bits is better.
- Factoring is just one line of attack
  - not necessarily the most practical
  - more attacks to take into account…

# Attacks against RSA

- Dictionary attack
  - If only two possible messages $m_0$ and $m_1$, then only two ciphertexts $c_0=m_0^e$ [n] and $c_1=m_1^e$ [n].
  - Encryption must be probabilistic (or non-static).
- Coppersmith's attack (1996)
  - Applies for RSA with small e, when some part of the message is known

# Attacks against RSA

- Chosen-ciphertext attack:

  Given ciphertext c to be decrypted
  - Generate a random r
  - Ask for the decryption of the random looking ciphertext $c' = c * (r^e)[n]$
  - One gets $m' = c'^d = c^d * (r^e)^d = c^d * r = m * r\ [n]$
  - This enables to compute $m = m'/r\ [n]$

# Attacks against RSA

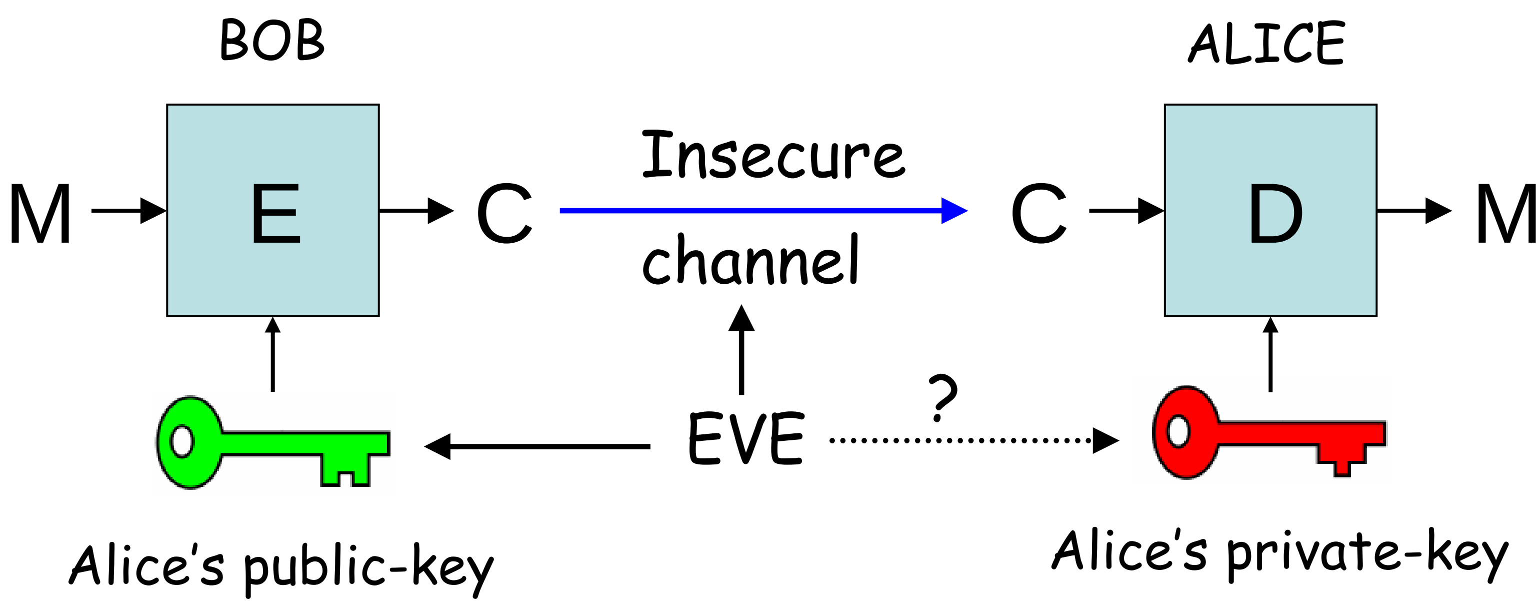
- One cannot use plain RSA encryption
  - one must add some randomness
  - one must apply some preformatting to the message
- Example: PKCS#1 v1.5
  - Encryption: $\mathbf{m}(m)=0002 \mid r \mid 00 \mid m$, then $c=\mathbf{m}(m)^d\,[n]$
  - Decryption: recover $\mathbf{m}(m)$, check redundancy.
- Bleichenbacher's attack against PKCS#1 v1.5
  - Appeared in 1998. Could be use against web-servers using SSL protocol.

# Security of RSA (and other cryptosystems)

- To be rigorous when speaking about security, one must specify
  - the attacker's goal:
    does he need to recover the key or only decrypt a particular ciphertext or less ?
  - the attacker's power:
    does he get only the user's public-key, or more ?

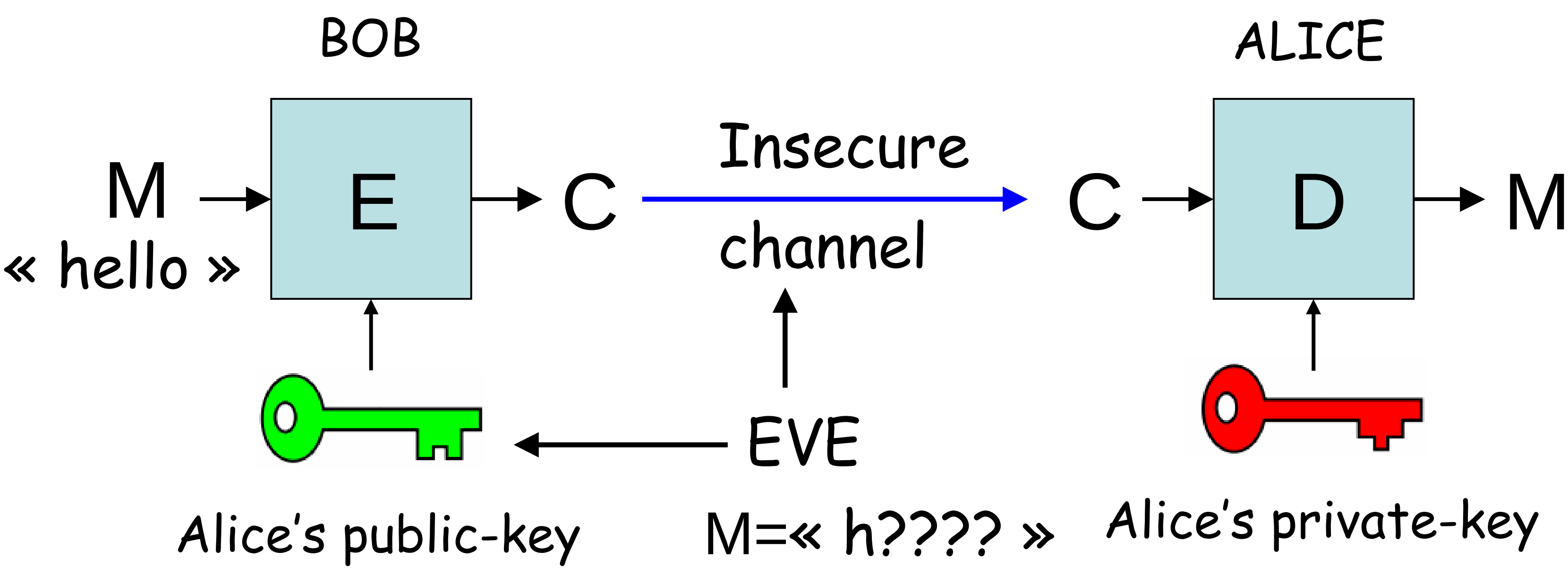
# Attacker's goal

- One may think that the adversary's goal is always to recover the private key.
  - complete break
  - may be too ambitious in practice



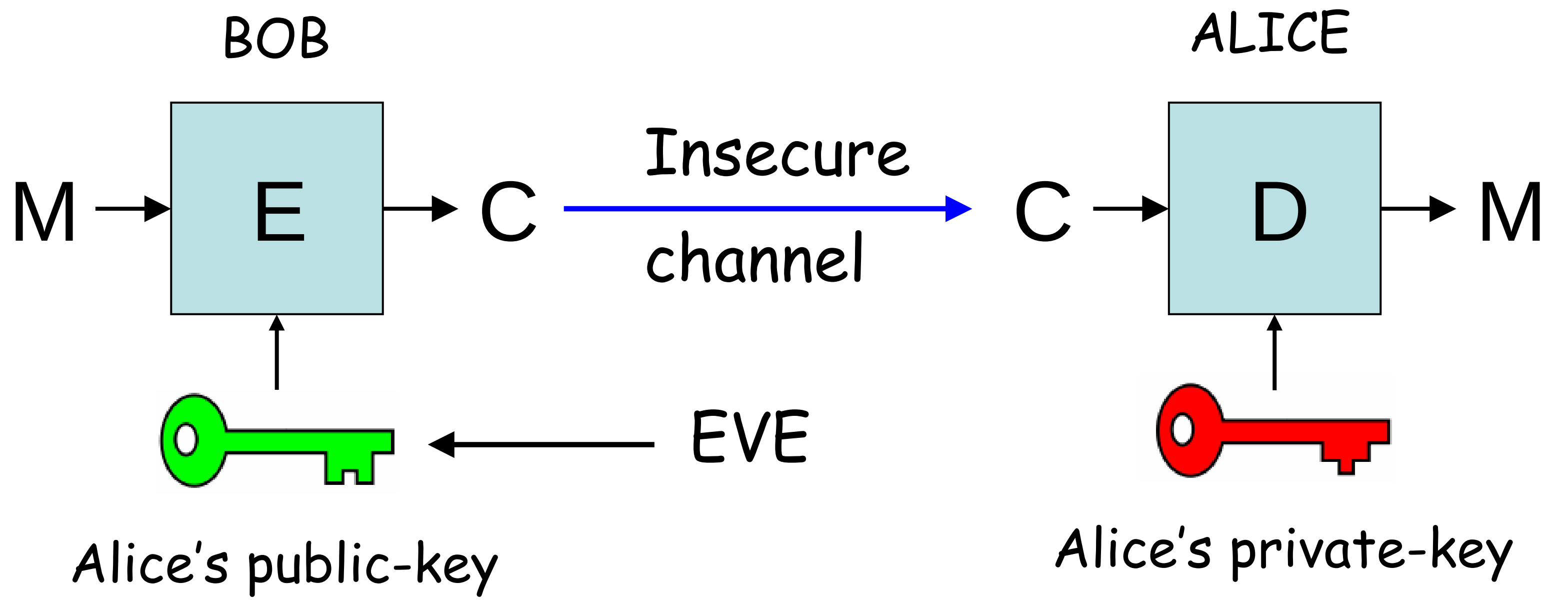
# Attacker's goal

- More modest goal: being able to decrypt one ciphertext.
  - or recover some information about a plaintext (for example, the first character)



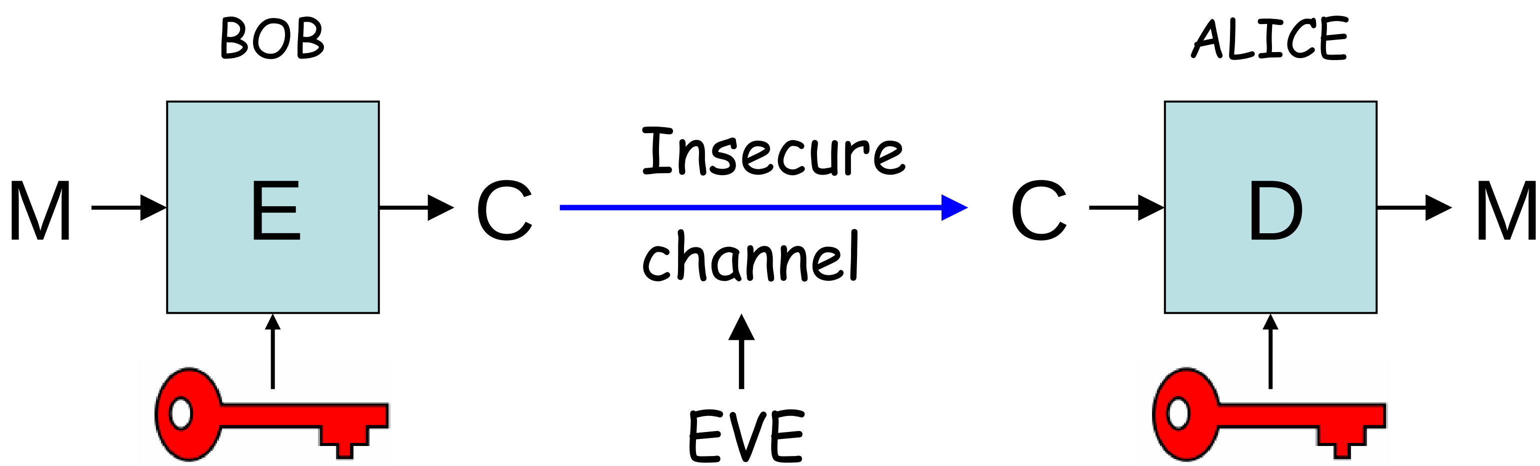
# Attack models

- Specify the power of the attacker
- Public-key only
  - the attacker gets only the public-key
  - Weakest adversary



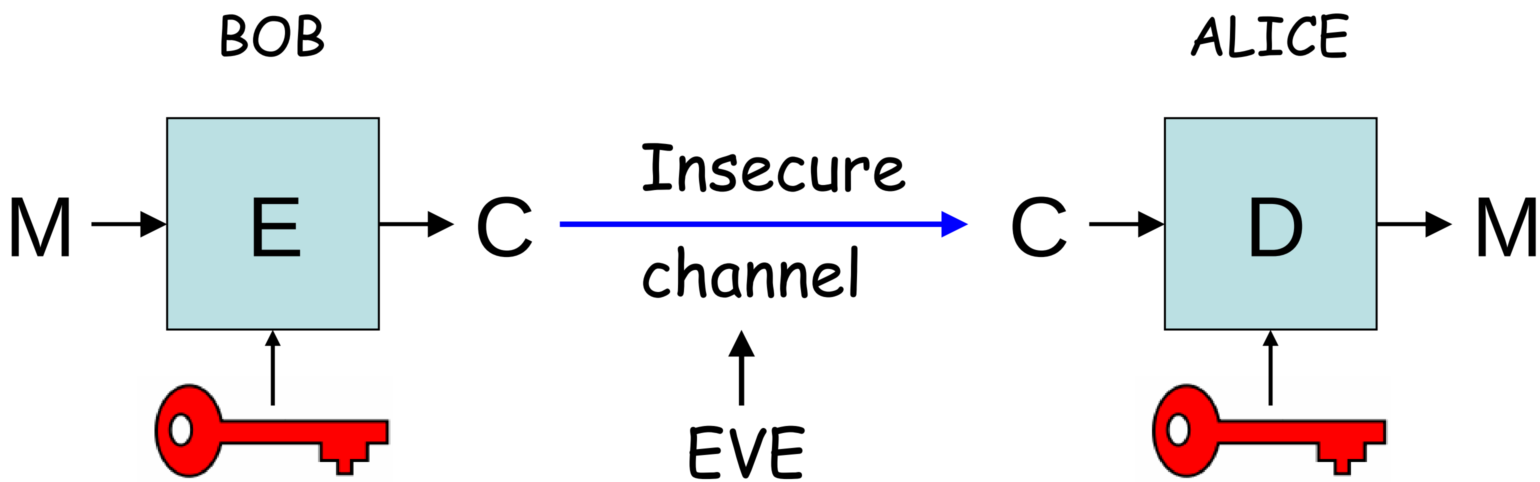
# Attack models

- Ciphertext-only attack
  - the attacker gets only a set of ciphertexts
  - primitive ciphers (Ceasar's cipher, mono-alphabetic substitution cipher) were vulnerable.



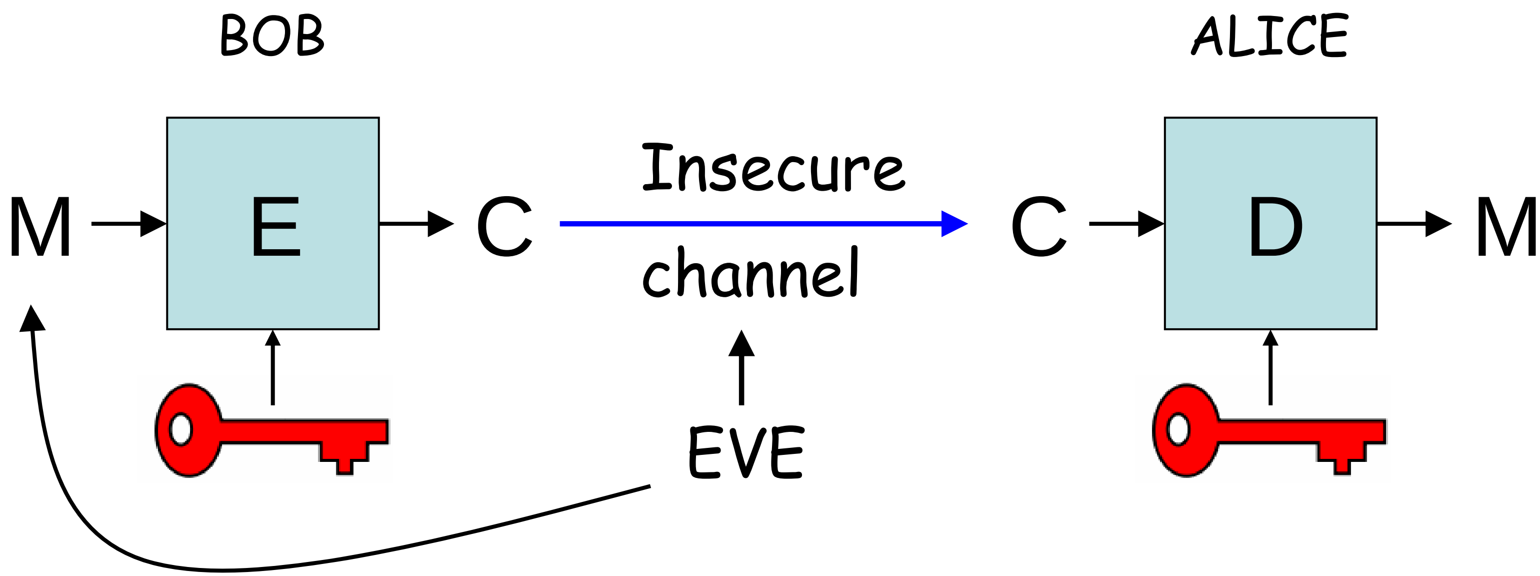
# Attack models

- Known-plaintext attack
  - Attack has access to plaintext/ciphertext pairs.
  - In practice, attacker may have some hint on some plaintexts.
  - Used during WW2 to break Enigma cipher.



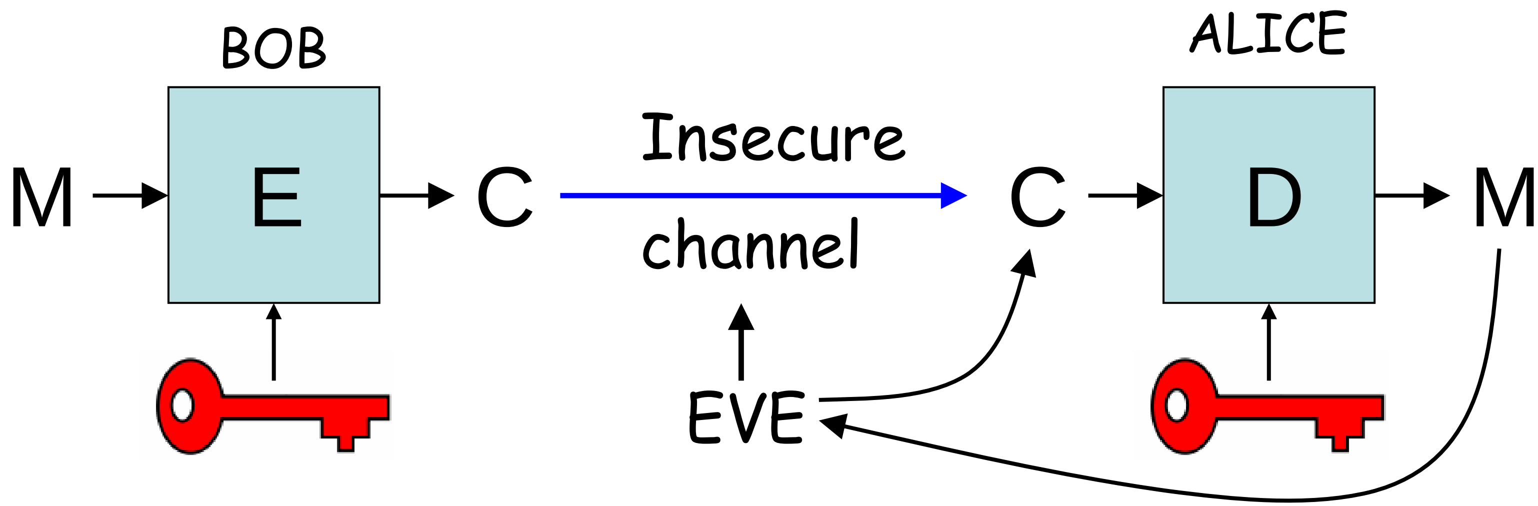
# Attack models

- Chosen plaintext attack
  - Attacker can obtain encryption of plaintexts of his choice.
  - For PK encryption, equivalent to PK only attack.



# Chosen-ciphertext attack

- Most powerful attack
- The attacker can obtain decryption of messages of his choice
- May be realistic in practice
  - attacker gets access to a decryption machine
  - encryption algorithm used in a more complex protocol in which users can obtain decryption of chosen ciphertexts.



# Attack scenario

- One must specify
  - the attacker's goal (total break, partial decryption…)
  - The attack model (chosen plaintext, chosen ciphertext…)
- Strongest security model: combines
  - weakest goal: obtaining only one bit of information about a plaintext
  - strongest adversary: chosen ciphertext attack

# Strongest security notion

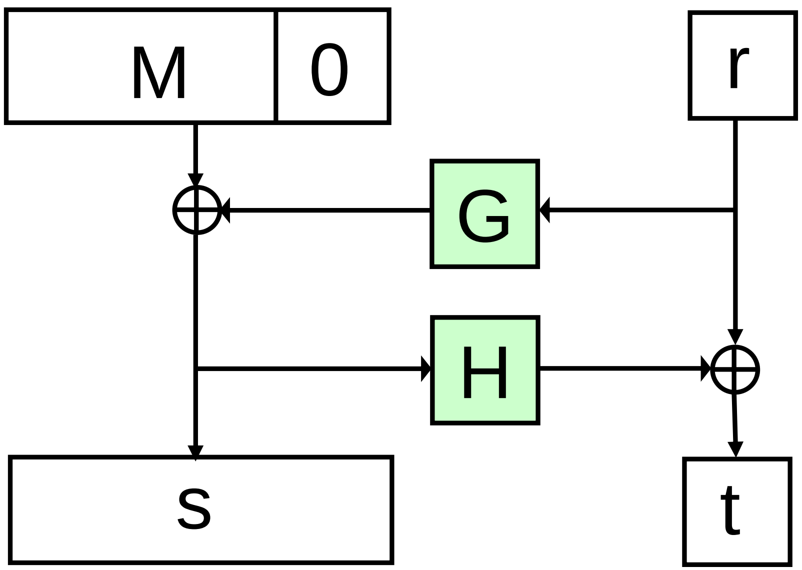
- Indistinguishability under adaptive chosen ciphertext attack (IND-CCA2)
  - Formalized in 1991 by Rackoff et Simon
  - *A* ciphertext should give no information about the corresponding plaintext, even under an adaptive chosen-ciphertext attack.
  - Has become standard security notion for encryption.

# IND-CCA2 schemes

- OAEP
  - Designed by Bellare and Rogaway in 1994.
  - Appears in PKCS#1 v2.1 standard.
- Cramer-Shoup (1998)
  - Based on discrete-log.
  - Proven secure without the random oracle model.

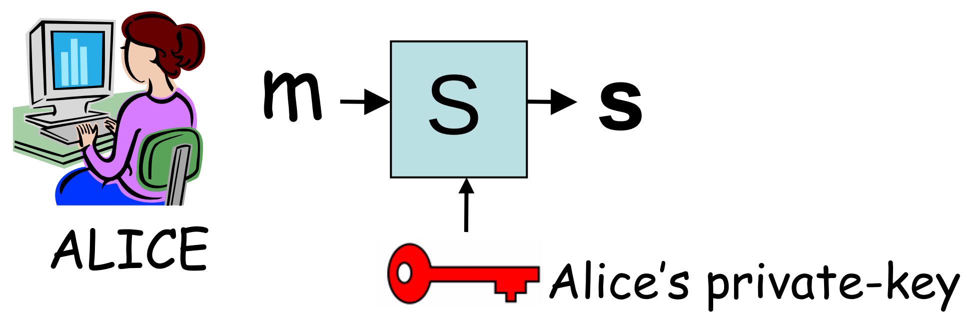
# OAEP

- Ciphertext is $c=(s|t)^e\ [n]$

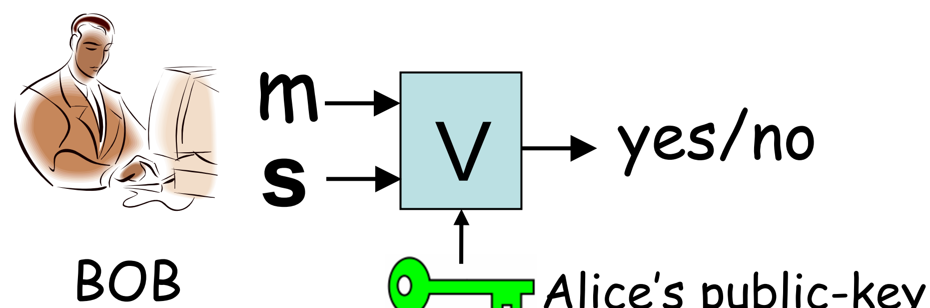


# Digital signature

- A bit string that depends on the message m and the user's public-key
  - Only Alice can sign a message using her private-key




  - Anybody can verify Alice's signature of m given her public-key



# Digital signature

- A digital signature provides:
  - Authenticity: only Alice can produce a signature of a message valid under her public-key.
  - Integrity: the signed message cannot be modified.
  - Non-repudiation: Alice cannot later claim that she did not sign the message

# Signing with RSA

- Public key: $n=p.q$ and $e$
- Private key: $d$ such that
  $$e.d=1 \bmod (p-1)(q-1)$$
- Signing using private $d$:
  $$\mathbf{s}=m^d \bmod n$$
- Verifying using public $n,e$:
  check that $m=\mathbf{s}^e \bmod n$
- ISO 9796-2, PKCS#1 v2.1

# Attacks against RSA signatures

- Given $\mathbf{s_1}=m_1^d \bmod n$ and $\mathbf{s_2}=m_2^d \bmod n$
  - one can compute the signature of $m_1*m_2$ without knowing $d$

    $\mathbf{s} = \mathbf{s_1}*\mathbf{s_2}=(m_1^d)*(m_2^d) \bmod n =(m_1*m_2)^d \bmod n$
- One cannot use plain RSA signature
  - One must apply some pre-formatting to the message to cancel the mathematical structure.

# RSA signature

- To prevent these attacks, one uses a hash function
  - PKCS#1 v1.5 :
    m(m)=0001 FF … FF00 ¦ c ¦ H(m)
  - ISO 9796-2:
    m(m)=6A ¦ m[1] ¦ H(m) ¦ BC

# Attack scenario for signature schemes

- We must specify
  - the adversary's goal
  - the adversary's power
- Adversary's goal
  - Controlled forgery: the adversary produces the signature of a message of his choice
  - Existential forgery: the adversary produces the signature of a (possibly meaningless) message

# Adversary's power

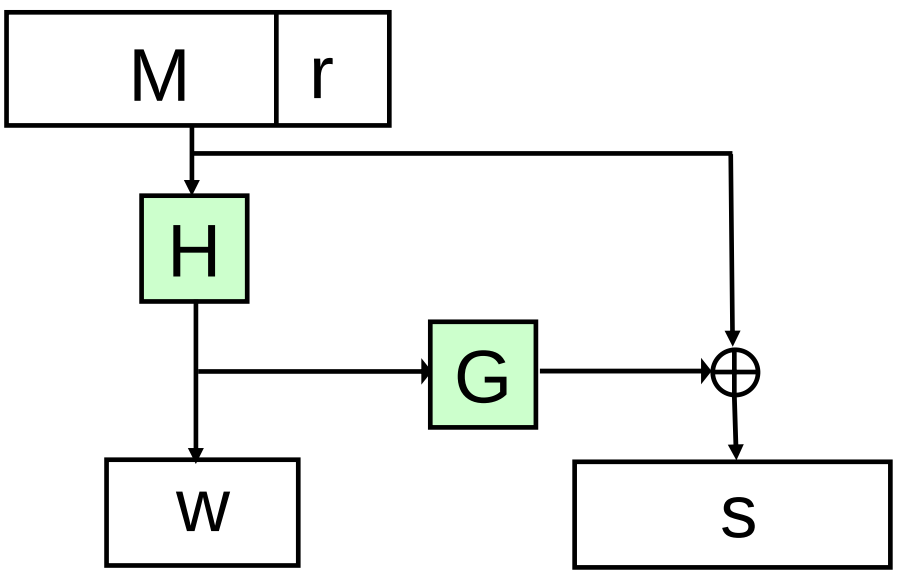
- No-message attack
  - The adversary gets only the public-key
- Known message attack
  - The adversary obtains a set of pairs message/signatures
- Chosen message attack
  - The adversary can obtain the signature of any message of his choice, adaptively.

# Strongest security notion

- Combines weakest goal with strongest adversary
- Existential unforgeability under an adaptive chosen message attack
  - Defined by Goldwasser, Micali and Rivest in 1988
  - It must be infeasible for an attacker to forge the signature of a message, even if he can obtain signature of messages of his choice.

# Example of secure signature schemes

- PSS
  - Designed by Bellare and Rogaway in 1996
  - IEEE P1363a standard and PKCS#1 v2.1
  - 2 variants: PSS and PSS-R that provides message recovery.




$\mathbf{s} = (w \,¦\, s)^{d} \bmod n$

# Conclusion

- What is cryptography ?
  - Cryptography's aim it to construct protocols that achieve some goal despite the presence of an adversary
- Scientific approach:
  - To be rigorous, one must define what it means to be secure
  - Then one tries to construct schemes that satisfies the definition, in a provable way.